Beltrami's Equation

MAT 550
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Abstract

In an expository setting, we discuss the existence and smoothness of solutions to the Beltrami equation which arise out of necessity for isothermal coordinates in Riemannian geometry. Topics in quasiconformal mapping theory due to Alfhors et. al are also discussed for formalism. The paper follows the expositions and research contained in [1], [2], [3], and [4].

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Motivation

The Beltrami equation arises from the need to prove the existence of isothermic coordinates; on a Riemannian manifold (M^n, g), they are local coordinates where the metric g is conformal to the Euclidean metric. In the case of a surface, for real valued u and v the condition that (u, v) be *isothermal* means that

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j = \lambda(du \otimes du + dv \otimes dv) \ , \ \lambda > 0.$$

Ideally, we would like to find a way to determine whether these coordinates exist on M^n. For starters, consider the dual metric g^* on $T^*\mathbb{R}^2$ which must satisfy

$$g^* = \sum_{i,j} g^{ij} \left(\frac{\partial}{\partial x^i}\right)^{**} \otimes \left(\frac{\partial}{\partial x^j}\right)^{**} = \frac{1}{\lambda} \left[\left(\frac{\partial}{\partial u}\right)^{**} \otimes \left(\frac{\partial}{\partial u}\right)^{**} + \left(\frac{\partial}{\partial v}\right)^{**} \otimes \left(\frac{\partial}{\partial v}\right)^{**} \right].$$

Setting (x^1, x^2) to (x, y) and the metric components $(g^{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and applying the above to the pairs $(du, du), (du, dv), (dv, dv)$ gives the two equations

$$a\left(\frac{\partial u}{\partial x}\right)^2 + 2b\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + c\left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{\lambda} = a\left(\frac{\partial v}{\partial x}\right)^2 + 2b\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + c\left(\frac{\partial v}{\partial y}\right)^2$$
$$a\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + b\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + c\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0.$$

After factoring the second equation this implies that there is a function ρ such that

$$\frac{\partial u}{\partial x} = \rho\left(b\frac{\partial v}{\partial x} + c\frac{\partial v}{\partial y}\right)$$
$$\frac{\partial u}{\partial y} = -\rho\left(a\frac{\partial v}{\partial x} + c\frac{\partial v}{\partial y}\right).$$

Substituting this into the first equation gives $\rho^2(ac - b^2) = 1$ and thus the Beltrami equations can be written as

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{ac - b^2}}\left(b\frac{\partial v}{\partial x} + c\frac{\partial v}{\partial y}\right) \ , \ \frac{\partial u}{\partial y} = -\frac{1}{\sqrt{ac - b^2}}\left(a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y}\right).$$

To simplify this data, it's common practice to set $f = u + iv$ and algebraic manipulations gives

$$\frac{\partial f/\partial \bar{z}}{\partial f/\partial z} = \frac{c - a - 2ib}{c + a + 2\sqrt{ac - b^2}} = \mu.$$

Notice that we always have

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2.$$

Hence if w satisfies the Beltrami equations, we can write

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left|\frac{\partial f}{\partial z}\right|^2 \left(1 - |\mu|^2\right).$$

This implies that the pair (u, v) has nonzero Jacobian whenever $\partial f/\partial z \neq 0$, since $|\mu| < 1$. Here we call μ the *complex dilatation* of w. The converse direction is also a simple calculation.

Quasi-conformal mappings

Now that we understand where this equation comes from, we need to understand what the solutions to it look like. In the literature there are three main definitions given, all of which are (provably) equivalent to each other.

Definition 2.0.1. A diffeomorphism $f : \Omega \to \Omega'$ is K-*quasiconformal* if f is orientation preserving and satisfies the inequality

$$\max_{\alpha} |\partial_\alpha f(z)| \leq K \min_{\alpha} |\partial_\alpha f(z)|$$

for every point $z \in \Omega$. Furthermore, for a family Γ of Jordan arcs/curves lying in Ω, let $P(\Gamma)$ be the set of Borel functions ρ such that $\int_\gamma \rho |dz| \geq 1$ for $\gamma \in \Gamma$. Then the *module* and *extremal length* of Γ are $\mathcal{M}(\Gamma)$ and $1/\mathcal{M}(\Gamma)$ respectively where

$$\mathcal{M}(\Gamma) = \inf_{\rho \in P(\Gamma)} \iint_\Omega \rho^2 \, dxdy.$$

From the definition, it's clear that if f is a K-quasiconformal diffeomorphism that $\mathcal{M}(f(\Gamma)) \leq KM(\Gamma)$. The converse to this statement, however, is of utmost importance. Rather than a family Γ, the converse holds when we associate f to families of quadrilaterals.

Definition 2.0.2. A *quadrilateral* $\mathcal{Q}$ is a Jordan domain equipped with a sequence of points $z_1, z_2, z_3, z_4 \in \partial\mathcal{Q}$. If $\mathcal{Q}$ is mapped conformally onto a rectangle $\mathcal{R}$ so that the points correspond to the vertices of $\mathcal{R}$, then the *module* of the quadrilateral $\mathcal{Q}$ is $\mod \mathcal{Q} = a/b$ where a is the length corresponding to the arc (z_1, z_2) and b to the arc (z_2, z_3).

Remark 2.0.3. It is clear that $\mod \mathcal{Q}$ is well-defined and conformally invariant.

Conformal invariant of the module is important because we can assume that $\mathcal{Q}$ is a rectangle in certain instances. If we do this by setting $z_1 = 0$, $z_2 = M > 0$, $z_3 = M + i$ and $z_4 = i$, then arithmetic shows that $\mod \mathcal{Q} = M$. Let Γ be a family of locally rectifiable arcs joining (z_1, z_2) and (z_3, z_4). Then for $\rho \in P(\Gamma)$ we have

$$1 \leq \left(\int_0^1 \rho(x+iy)\, dy \right)^2 \leq \int_0^1 \rho^2(x+iy)\, dy \implies M \leq \iint_{\mathcal{Q}} \rho^2 \, dxdy \implies \mod \mathcal{Q} = \mathcal{M}(\Gamma).$$

Hence the above discussion reveals the idea that if f is a K-quasiconformal diffeomorphism of Ω, the fact that $\mathcal{M}(f(\Gamma)) \leq K\mathcal{M}(\Gamma)$ and the above equality shows that

$$\mod f(\mathcal{Q}) \leq K \mod \mathcal{Q}$$

for every quadrilateral $\mathcal{Q}$ with closure $\overline{\mathcal{Q}} \subseteq \Omega$. In particular, the above inequality holds if and only if f is K-quasiconformal. In a more general form, we have our main definition that we will use repeatedly.

Definition 2.0.4. A homeomorphism $f : \Omega \to \Omega'$ is K-quasiconformal if f is orientation preserving and satisfies the inequality

$$\mod f(\mathcal{Q}) \leq K \mod \mathcal{Q}$$

for all quadrilaterals $\mathcal{Q}$ with closure $\overline{\mathcal{Q}} \subseteq \Omega$.

What is nice about K-quasiconformality is that it's a local property. If f is K-quasiconformal in a neighborhood of every point, then it's K-quasiconformal everywhere in Ω.

Definition 2.0.5. We shall say that a complex-valued function f is *absolutely continuous on lines* in the region Ω if for every closed rectangle $\mathcal{R} \subseteq \Omega$ with sides parallel to the x and y-axes, f is absolutely continuous almost everywhere on horizontal and vertical lines in $\mathcal{R}$.

An involved argument shows that our previous definition of K-quasiconformality is equivalent to the following theorem, which allows us to approach the concept of K-quasiconformality from a purely analytic perspective.

Theorem 2.0.6. *An orientation preserving homeomorphism f is K-quasiconformal if and only if f is absolutely continuous on lines and the inequality holds*

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq \frac{K-1}{K+1}\left|\frac{\partial f}{\partial z}\right|.$$

The payoff of all of the work we did above can be properly formalized in a theorem, which we will use as the foreground for the analytical lemmas needed to prove the existence of solutions to Beltrami's equation.

Theorem 2.0.7. *Let f be a K-quasiconformal mapping. Then f is differentiable almost everywhere, satisfies the inequality*

$$\max_{\alpha} |\partial_\alpha f(z)| \leq K \min_{\alpha} |\partial_\alpha f(z)|$$

almost everywhere and its derivatives are locally L^2-integrable.

Proof. Since f is almost continuous on lines, its partial derivatives exist almost eveywhere and differentiability almost everywhere is implied. The second statement holds trivially from the previous discussion, so we prove the third statement. Consider the equation

$$\partial_\alpha f(z) = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \cdot e^{-2i\alpha} \tag{$\star$}$$

From this we can determine that the extrema below

$$\max |\partial_\alpha f(z)| = \left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \bar{z}}\right|$$
$$\min |\partial_\alpha f(z)| = \left|\frac{\partial f}{\partial z}\right| - \left|\frac{\partial f}{\partial \bar{z}}\right|$$

hold. Hence multiplying both sides of ($\star$) by $\max |\partial_\alpha f(z)|$ we obtain $\max |\partial_\alpha f(z)|^2 \leq KJ(z)$ where J is the Jacobian of f, and since the Jacobian of an almost everywhere differentiable homeomorphism is locally integrable the desired is achieved. ∎

We finish this portion of our work with a short dicussion on generalized derivatives. For a complex valued f in a domain $\Omega \subseteq \mathbb{C}$, we say that f has *generalized L^p derivatives* in Ω if f is almost continuous on lines and the partial derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ belong locally to L^p. The following theorem states how such functions behave.

Theorem 2.0.8. *Let f, g and h be functions in Ω, with g and h belonging locally to L^p. Let $\{f_n\}_{n=1}^{\infty} \in C_0^{\infty}$ be such that for every compact subset $A \subset \Omega$*

(1) The sequence f_n converges uniformly to f on A as $n \to \infty$.

(2) The integrals

$$\lim_{n\to\infty} \iint \left|\frac{\partial f_n}{\partial z} - g\right|^p dxdy = \lim_{n\to\infty} \iint \left|\frac{\partial f_n}{\partial \bar{z}} - h\right|^p dxdy = 0.$$

Then f has generalized L^p derivatives in Ω. The converse also holds as well with $g = \partial f/\partial z$ and $h = \partial f/\partial \bar{z}$.

The point of mentioning this result is that we want to quickly generalize Green's formula using the latter part of this theorem. Let f and g with generalized L^p and L^q derivatives in Ω such that $1/p + 1/q = 1$. Then if X is a Jordan domain with $\overline{X} \subseteq \Omega$ with rectifiable boundary on which g has a bounded variation, then

$$\int_{\partial X} f\,dg = -2i \iint_X \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}}\frac{\partial g}{\partial z} \right) dxdx$$

where ∂X is positively oriented. By setting $g(z) = z$ and $g(z) = \bar{z}$ for example, this allows us to set

$$\int_{\partial X} f\,dz = 2i \iint_X \frac{\partial f}{\partial \bar{z}}\,dxdy$$
$$\int_{\partial X} f\,d\bar{z} = -2i \iint_X \frac{\partial f}{\partial z}\,dxdy$$

which we will make use of shortly.

The Equation

At this point we now seek to show the existence and smoothness of the solutions to the Beltrami equation. Using what we have so far, Theorem 2 and the equivalence of the K-quasicovariance definitions allow us to arrive at the following preliminary result.

Theorem 3.0.1. *Let f be an orientation preserving homeomorphism with complex dilatation factor μ such that $||\mu||_\infty < 1$. Then f has generalized L^2 derivatives and the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$$

holds almost everywhere. Conversely, if μ is a measureable function satisfying the Beltrami equation almost eveywhere, then every such f is a K-quasiconformal mapping.

Hence our labors are justified in the sense that every solution to the Beltrami equation is a K-quasiconformal mapping. Our goal now is to somehow right f in terms of the complex dilatation factor μ. To get started, let f be a function with L^1 derivatives in a domain Ω. Let A be a Jordan domain with rectifiable boundary, such that $\overline{A} \subseteq \Omega$ and $\overline{A_r} \subseteq \Omega$ where $A_r = \{\zeta \ : \ |\zeta - z| < r\}$. Define the function

$$\psi(\zeta) = \begin{cases} \frac{f(\zeta)}{\zeta - z}, & \zeta \in \Omega \backslash A_r \\ \frac{f(\zeta)(\bar{\zeta} - \bar{z})}{r^2}, & \zeta \in A_r. \end{cases}$$

Using our previous work from Green's theorem, we can use the formula below

$$\int_{\partial A} f \, dz = 2i \iint_A \frac{\partial f}{\partial \bar{z}} \, dxdy$$

and apply it to ψ on A and A_r. Subtracting and sending $r \to 0$ gives the Cauchy-Pompeiu formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial A} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{\pi} \iint_A \frac{\partial f / \partial \bar{z}}{\zeta - z} \, d\xi d\eta$$

where $\zeta = \xi + i\eta$. Now suppose that f has L^1 derivatives in all of $\mathbb{C}$ and $f(z) \to 0$ as $z \to \infty$. By defining

$$T(\omega(z)) := -\frac{1}{\pi} \iint \frac{\omega(\zeta)}{\zeta - z} \, d\xi d\eta$$

over $\mathbb{C}$, from Cauchy-Pompeiu we immediately have $f = T(\partial f / \partial \bar{z})$. If $\omega \in C_0^\infty(\mathbb{C})$, we can differentiate $T(\omega(z))$ and obtain

$$\frac{\partial T(\omega)}{\partial \bar{z}} = \omega \ , \ \frac{\partial T(\omega)}{\partial z} = S(\omega(z)) := -\frac{1}{\pi} \iint \frac{\omega(\zeta)}{(\zeta - z)^2} \, d\xi d\eta.$$

This operator S is called the *Hilbert transformation*, and it's worth noting that $T, S \in C^\infty(\mathbb{C})$. Before we move on, we state a theorem without proof.

Lemma 1 (Calderón-Zygmund)**.** *S is bounded in $L^p \cap C_0^\infty(\mathbb{C})$ for every $p > 1$. In other words, there is a constant c depending only on p such that*

$$||S(\omega(z))||_p \leq c_p ||\omega(z)||_p.$$

Because S preserves the L^2-norms of $C_0^\infty(\mathbb{C})$ functions, the Calderón-Zygmund will allow us to alleviate issues in isometry relations in future theorems we prove. More importantly, we also have the following

Lemma 2 (Riesz-Thorin). *The best constant c_p is such that $\log(c_p)$ is a convex function of $1/p$. In other words c_p is continuous in p.*

Using both of these theorems, we are able to write f in terms of the complex dilatation factor. The main idea is to write it in terms of a series expansion, and to justify convergence. We do this concisely in the theorem stated next.

Theorem 3.0.2. *Let f be a K-quasiconformal mapping in $\mathbb{C}$ with complex dilitation factor μ. If μ has bounded support and $f(z) - z \to 0$ as $z \to \infty$ then*

$$f(z) = z + \sum_{n=1}^{\infty} T(\phi_n(z))$$

where $\{\phi_n(z)\}_{n=1}^{\infty}$ are defined by $\phi_1(z) = \mu$ and $\phi_n = \mu S(\phi_{n-1}(z))$.

Proof. Because $f = T(\partial f/\partial \bar{z})$ as remarked earlier, consider $z \mapsto f(z) - z$. Then $f(z) = z + T(\partial f/\partial \bar{z})$ and differentiating with respect to z gives

$$\frac{\partial f}{\partial z} = 1 + S\left(\frac{\partial f}{\partial \bar{z}}\right)$$

almost everywhere. To justify this, there are two cases. If $\partial f/\partial \bar{z} \in C_0^{\infty}(\mathbb{C})$, then this follows from the derivative relation between T and S. More generally this holds because f has L^2 general derivatives by Theorem 2. By Theorem 3 we know that the Beltrami equation holds almost everywhere, so the above tells us that $\partial f/\partial \bar{z}$ must satisfy

$$\omega(z) = \mu + \mu S(\omega(z))$$

almost everywhere. Now if $||\mu||_{\infty} = K$, the Calderón-Zygmund inequality (Lemma 3) says that for $\phi_n(z)$ defined previously

$$||\phi_{n+1}(z)||_p \leq Kc_p||\phi_n(z)||_p \leq \ldots \leq (Kc_p)^n||\mu||_p.$$

Now choose $p > 2$. By Lemma 3 we know that c_p is continuous and $c_2 = 1$ since S preserves L^2-norms of $C_0^{\infty}(\mathbb{C})$ functions. Hence we can consider $Kc_p < 1$, and recursively write

$$\omega_n(z) = \sum_{i=1}^{n} \phi_i(z).$$

By our work, this is convergent in L^p. Moreover $\omega_{n+1}(z) = \mu + \mu S\omega_n(z)$ so its limit ω satisfies the previous relation. Now if ψ is another solution, then

$$||\omega - \psi||_p \leq Kc_p||\omega - \psi||_p$$

and hence $\psi \equiv \omega$ almost everywhere. Hence $\omega_n \to \partial f/\partial \bar{z}$ in L^p. Using Hölder's inequality we can estimate $|T(\omega(z))| \leq C||\omega(z)||_p$, and hence the fact that $\frac{\partial f}{\partial z} = 1 + S\left(\frac{\partial f}{\partial \bar{z}}\right)$ gives us the theorem and this series converges uniformly and absolutely. ■

We are now able to prove the existence of solutions to Beltrami's equation, and we waste no time at all in doing so. We present it of course as a theorem.

Theorem 3.0.3 (Morrey-Alfors-Bers). *Let μ be a measurable function in a domain Ω with $||\mu||_{\infty} < 1$. Then there exists a K-quasiconformal mapping (with $1 \leq K < \infty$) of Ω whose complex dilitation factor agrees with μ almost everywhere.*

Proof. We first show that if $\mu \in C_0^{\infty}(\mathbb{C})$ that the equation

$$\frac{\partial h}{\partial \bar{z}} = \mu\frac{\partial h}{\partial z}$$

has a locally injective solution. Now consider the non-homogenous version of the Beltrami equation which we write as

$$\frac{\partial h}{\partial \bar{z}} = \mu \frac{\partial h}{\partial z} + \frac{\partial \mu}{\partial z}.$$

From the methods to prove the previous result we can consider the equation

$$\omega(z) = S\left(\frac{\partial \mu}{\partial z}\right) + S(\mu\omega(z)).$$

As usual set $\phi_1 = S(\partial\mu/\partial z)$ and $\phi_n = S(\mu\phi_{n-1}(z))$ for $n \geq 2$, and by previous methods we can choose $p > 2$ such that $c_p||\mu||_\infty < 1$. Then we have

$$||\phi_{n+1}(z)||_p \leq (c_p||\mu||_\infty)^n \left|\left|\frac{\partial \mu}{\partial z}\right|\right|_p.$$

Hence by the above reasoning from before we can write

$$\omega_n(z) = \sum_{i=1}^{n} \phi_i(z)$$

and this converges in L^p to a limit which is the desired solution $\omega(z)$. We show that

$$h(z) = T\left(\frac{\partial h}{\partial z} + \mu\omega(z)\right)$$

has L^p derivatives and satisfies the above non-homogenous ODE almost everywhere. Choose $\omega_n \in C_0^\infty(\mathbb{C})$ so that $||\omega - \omega_n||_p \to 0$ as $n \to \infty$. Now for f_n defined by

$$f_n(z) = T\left(\frac{\partial h}{\partial z} + \mu\omega_n(z)\right),$$

it follows from the differentiability relation between T and S that

$$\frac{\partial f_n}{\partial z} = S\left(\frac{\partial \mu}{\partial z} + \mu\omega_n(z)\right), \quad \frac{\partial f_n}{\partial \bar{z}} = \frac{\partial \mu}{\partial z} + \mu\omega_n(z).$$

Now we apply Theorem 2 this proves the result for h. By the generalized Green's theorem we can pick some $f \in C^1(\mathbb{C})$ such that

$$f(z) = \int_0^z e^{h(z)}(dt + \mu(t))\, d\bar{t}$$

where $\partial f/\partial z = \exp(h(z))$ and $\partial f/\partial \bar{z} = \mu \exp(h(z))$. Hence it is a solution to the original Beltrami equation, and the Jacobian

$$J(z) = |\exp(h(z))|^2(1 - |\mu|^2) > 0$$

implies that this solution is locally injective. Next we prove that this solution is globally injective using a topological argument. Note that f is a homeomorphism of the plane. Because $\partial\mu/\partial z + \mu\omega(z)$ has bounded support, we can conclude that $h(z) \to 0$ as $z \to \infty$ which means that $\partial f/\partial z = e^{h(z)} \to 1$ in the limit. Using the above integral relation for f, we see that $f \to \infty$ at ∞ and thus f maps the plane Ω to itself. If not, then there are points z_n such that $f(z_n)$ converges to a finite boundary point of the image. But the points z_n can't accumulate at ∞, contradiction. We also see that from this argument no point can have infinitely many preimages, so every point has an open neighborhood V onto which each component of $f^{-1}(V)$ maps topologically. Hence (f, Ω) is a universal covering surface of Ω, and since Ω is simply connected f is a homeomorphism of Ω by the Monodromy theorem.

Remark 3.0.4. Note that this topological result follows also from an application of the uniformization theorem.

We finish the proof by approximating any arbitrary μ by $C_0^\infty(\mathbb{C})$ functions to obtain the general solution. Let f_n be μ_n-quasiconformal self-mappings of the finite plane, with $f_n(0) = 0$ and $f_n(1) = 1$ and $||\mu_n|| \leq k < 1$. If $\mu_n \to \mu$ almost everywhere, then there exists (by a corollary, see Lehto) a subsequence of f_n which converges to a μ_n-quasiconformal mapping of the plane. Moreover, by a theorem which we use unproved the mappings f_n constitute a normal family (a precompact subspace of the space of continuous functions). Thus, a subsequence exists which is uniformly convergent in every compact subset of the plane. This normalization guarantees that $f_n \to f$ is a K-quasiconformal map. Since $\mu_n \to \mu$ almost everywhere, by an unproven lemma we have that f is μ-quasiconformal.

Now let μ be an arbitrary measurable function in the plane with $||\mu||_\infty < 1$ of course. Then there are $C_0^\infty(\mathbb{C})$ functions $\mu_n \to \mu$ almost everywhere with $||\mu_n||_\infty \leq ||\mu||_\infty$. By the global injective solution constructed, we know that there exist μ_n-quasiconformal mappings f_n in the plane, and we have just shown that we can normalize them so that $f = \lim f_n$ is μ-quasiconformal. We have thus produced the desired mapping. ■

Now that we have proven existence, it follows from similar arguments that the derivatives of a K-quasiconformal mapping are locally in a class L^p for some $p > 2$. The key ingredient to the proof is simply observing that if f and g are quasiconformal mappings for (possibly) differing constants with $\mu_f(z) = \mu_g(z)$, then $g \circ f^{-1}(z)$ is a conformal map. This, ultimately is just a computation.

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